

# Book drawings of complete bipartite graphs

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## Abstract

We recall that a *book with  $k$  pages* consists of a straight line (the *spine*) and  $k$  half-planes (the *pages*), such that the boundary of each page is the spine. If a graph is drawn on a book with  $k$  pages in such a way that the vertices lie on the spine, and each edge is contained in a page, the result is a  *$k$ -page book drawing* (or simply a  *$k$ -page drawing*). The *pagenumber* of a graph  $G$  is the minimum  $k$  such that  $G$  admits a  $k$ -page embedding (that is, a  $k$ -page drawing with no edge crossings). The  *$k$ -page crossing number*  $\nu_k(G)$  of  $G$  is the minimum number of crossings in a  $k$ -page drawing of  $G$ . We investigate the pagenumbers and  $k$ -page crossing numbers of complete bipartite graphs. We find the exact pagenumbers of several complete bipartite graphs, and use these pagenumbers to find the exact  $k$ -page crossing number of  $K_{k+1,n}$  for  $k \in \{3, 4, 5, 6\}$ . We also prove the general asymptotic estimate  $\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \nu_k(K_{k+1,n}) / (2n^2/k^2) = 1$ . Finally, we give general upper bounds for  $\nu_k(K_{m,n})$ , and relate these bounds to the  $k$ -planar crossing numbers of  $K_{m,n}$  and  $K_n$ .

**Keywords:** 2-page crossing number, book crossing number, complete bipartite graphs, Zarankiewicz conjecture

**AMS Subject Classification:** 90C22, 90C25, 05C10, 05C62, 57M15, 68R10

## 1 Introduction

In [5], Chung, Leighton, and Rosenberg proposed the model of embedding graphs in books. We recall that a *book* consists of a line (the *spine*) and  $k \geq 1$  half-planes (the *pages*), such that the boundary of each page is the spine. In a *book embedding*, each edge is drawn on a single page, and no edge crossings are allowed. The *pagenumber* (or *book thickness*)  $p(G)$  of

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26 a graph  $G$  is the minimum  $k$  such that  $G$  can be embedded in a  $k$ -page book [2, 5, 11, 13].  
 27 Not surprisingly, determining the pagewidth of an arbitrary graph is NP-Complete [5].

28 In a *book drawing* (or *k-page drawing*, if the book has  $k$  pages), each edge is drawn on a  
 29 single page, but edge crossings are allowed. The *k-page crossing number*  $\nu_k(G)$  of a graph  
 30  $G$  is the minimum number of crossings in a  $k$ -page drawing of  $G$ .

31 Instead of using a straight line as the spine and halfplanes as pages, it is sometimes con-  
 32 venient to visualize a  $k$ -page drawing using the equivalent *circular model*. In this model,  
 33 we view a  $k$ -page drawing of a graph  $G = (V, E)$  as a set of  $k$  circular drawings of graphs  
 34  $G^{(i)} = (V, E^{(i)})$  ( $i = 1, \dots, k$ ), where the edge sets  $E^{(i)}$  form a  $k$ -partition of  $E$ , and such  
 35 that the vertices of  $G$  are arranged identically in the  $k$  circular drawings. In other words,  
 36 we assign each edge in  $E$  to exactly one of the  $k$  circular drawings. In Figure 1 we illustrate  
 37 a 3-page drawing of  $K_{4,5}$  with 1 crossing.

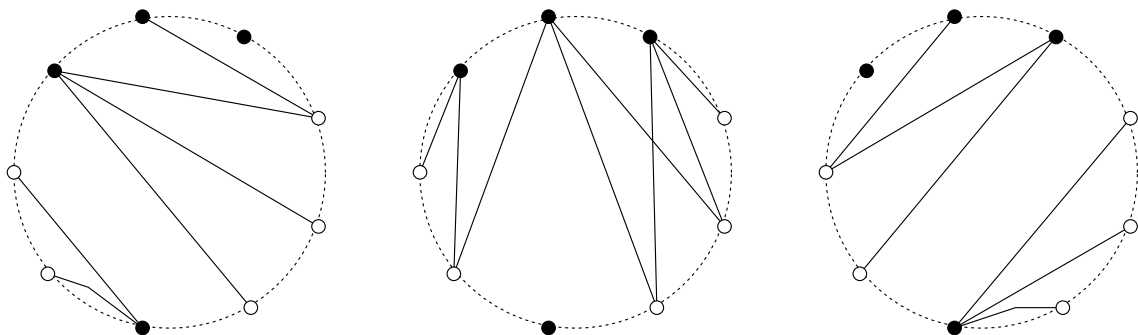


Figure 1: A 3-page drawing of  $K_{4,5}$  with 1 crossing. Vertices in the chromatic class of size 4 are black, and vertices in the chromatic class of size 5 are white. We have proved (Theorem 3) that the pagewidth of  $K_{4,5}$  is 4, and so it follows that  $\nu_3(K_{4,5}) \geq 1$ . Now this 1-crossing drawing implies that  $\nu_3(K_{4,5}) \leq 1$ , and so it follows that  $\nu_3(K_{4,5}) = 1$ .

38 Very little is known about the pagewidths or  $k$ -page crossing numbers of interesting fam-  
 39 ilies of graphs. Even computing the pagewidth of planar graphs is a nontrivial task;  
 40 Yannakakis proved [24] that four pages always suffice, and sometimes are required, to em-  
 41 bed a planar graph. It is a standard exercise to show that the pagewidth  $p(K_n)$  of the  
 42 complete graph  $K_n$  is  $\lceil n/2 \rceil$ . Much less is known about the  $k$ -page crossing numbers of  
 43 complete graphs. A thorough treatment of  $k$ -page crossing numbers (including estimates  
 44 for  $\nu_k(K_n)$ ), with general lower and upper bounds, was offered by Shahrokhi et al. [21].  
 45 In [9], de Klerk et al. recently used a variety of techniques to compute several exact  $k$ -page  
 46 crossing numbers of complete graphs, as well as to give some asymptotic estimates.

47 Bernhart and Kainen [2] were the first to investigate the pagewidths of complete bipartite  
 48 graphs, giving lower and upper bounds for  $p(K_{m,n})$ . The upper bounds in [2] were then  
 49 improved by Muder, Weaver, and West [18]. These upper bounds were further improved by  
 50 Enomoto, Nakamigawa, and Ota [12], who derived the best estimates known to date. Much  
 51 less is known about the  $k$ -page crossing number of  $K_{m,n}$ .

## 52 1.1 1-page drawings of $K_{m,n}$

53 Although calculating the 1-page crossing number of the complete graph  $K_n$  is trivial, this  
 54 is by no means the case for the complete bipartite graph  $K_{m,n}$ . Still, our knowledge about  
 55  $\nu_1(K_{m,n})$  is almost completely satisfactory, due to the following result by Riskin [20].

56 **Theorem 1** (Riskin [20]). *If  $m|n$  then  $\nu_1(K_{m,n}) = \frac{1}{12}n(m-1)(2mn-3m-n)$ , and this*  
 57 *minimum value is attained when the  $m$  vertices are distributed evenly amongst the  $n$  vertices.*

## 58 1.2 2-page drawings of $K_{m,n}$

59 Zarankiewicz's Conjecture states that the (usual) crossing number  $\text{cr}(K_{m,n})$  of  $K_{m,n}$  equals  
 60  $Z(m, n) := \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ , for all positive integers  $m, n$ . Zarankiewicz [25] found  
 61 drawings of  $K_{m,n}$  with exactly  $Z(m, n)$  crossings, thus proving  $\text{cr}(K_{m,n}) \leq Z(m, n)$ . These  
 62 drawings can be easily adapted to 2-page drawings (without increasing the number of cross-  
 63 ings), and so it follows that  $\nu_2(K_{m,n}) \leq Z(m, n)$ .

64 Since  $\text{cr}(G) \leq \nu_2(G)$  for any  $G$ , Zarankiewicz's Conjecture implies the (in principle, weaker)  
 65 conjecture  $\nu_2(K_{m,n}) = Z(m, n)$ . Zarankiewicz's Conjecture has been verified (for  $\text{cr}(K_{m,n})$ ,  
 66 and thus also for  $\nu_2(K_{m,n})$ ) for  $\min\{m, n\} \leq 6$  [14], and for the special cases  $(m, n) \in$   
 67  $\{(7, 7), (7, 8), (7, 9), (7, 10), (8, 8), (8, 9), (8, 10)\}$  [23]. Recently, de Klerk and Pasechnik [8]  
 68 used semidefinite programming techniques to prove that  $\lim_{n \rightarrow \infty} \nu_2(K_{7,n})/Z(7, n) = 1$ .

## 69 1.3 $k$ -page drawings of $K_{m,n}$ for $k \geq 3$ : lower bounds

70 As far as we know, neither exact results nor estimates for  $\nu_k(K_{m,n})$  have been reported in  
 71 the literature, for any  $k \geq 3$ . Indeed, all the nontrivial results known about  $\nu_k(K_{m,n})$  are  
 72 those that can be indirectly derived from the thorough investigation of Shahrokhi, Sýkora,  
 73 Székely, and Vrt'o on multiplanar crossing numbers [22].

74 We recall that a multiplanar drawing is similar to a book drawing, but involves unrestricted  
 75 planar drawings. Formally, let  $G = (V, E)$  be a graph. A  $k$ -planar drawing of  $G$  is a set of  
 76  $k$  planar drawings of graphs  $G^{(i)} = (V, E^{(i)})$  ( $i = 1, \dots, k$ ), where the edge sets  $E^{(i)}$  form a  
 77  $k$ -partition of  $E$ . Thus, to obtain the  $k$ -planar drawing, we take the drawings of the graphs  
 78  $G^{(i)}$ , and (topologically) identify the  $k$  copies of each vertex. The  $k$ -planar crossing number  
 79  $\text{cr}_k(G)$  of  $G$  is the minimum number of crossings in a  $k$ -planar drawing of  $G$ . A *multiplanar*  
 80 *drawing* is a  $k$ -planar drawing for some positive integer  $k$ .

81 It is very easy to see that  $\nu_k(G) \geq \text{cr}_{\lceil k/2 \rceil}(G)$ , for every graph  $G$  and every nonnegative  
 82 integer  $k$ . Thus lower bounds of multiplanar (more specifically,  $r$ -planar) crossing numbers  
 83 immediately imply lower bounds of book (more specifically,  $2r$ -page) crossing numbers. A  
 84 strong result by Shahrokhi, Sýkora, Székely, and Vrt'o is the exact determination of the

85  $r$ -planar crossing number of  $K_{2r+1,n}$  ([22, Theorem 3]):

$$\text{cr}_r(K_{2r+1,n}) = \left\lfloor \frac{n}{2r(2r-1)} \right\rfloor \left( n - r(2r-1) \left( \left\lfloor \frac{n}{2r(2r-1)} \right\rfloor - 1 \right) \right).$$

86 Using this result and our previous observation  $\nu_k(G) \geq \text{cr}_{\lceil k/2 \rceil}(G)$ , one obtains:

87 **Theorem 2** (Follows from [22, Theorem 3]). *For every even integer  $k$  and every integer  $n$ ,*

$$\nu_k(K_{k+1,n}) \geq \left\lfloor \frac{n}{k(k-1)} \right\rfloor \left( n - \frac{k}{2} \binom{k-1}{2} \left( \left\lfloor \frac{n}{k(k-1)} \right\rfloor - 1 \right) \right). \quad \square$$

88 Regarding general lower bounds, using the following inequality from [22, Theorem 5]

$$\text{cr}_r(K_{m,n}) \geq \frac{1}{3(3r-1)^2} \binom{m}{2} \binom{n}{2}, \quad \text{for } m \geq 6r-1 \text{ and } n \geq \max\{6r-1, 2r^2\},$$

89 and the observation  $\nu_k(K_{m,n}) \geq \text{cr}_{\lceil k/2 \rceil}(K_{m,n})$ , one obtains

$$\nu_k(K_{m,n}) \geq \frac{1}{3(3\lceil \frac{k}{2} \rceil - 1)^2} \binom{m}{2} \binom{n}{2}, \text{ for } m \geq 6\lceil k/2 \rceil - 1 \text{ and } n \geq \max\{6\lceil k/2 \rceil - 1, 2\lceil k/2 \rceil^2\}. \quad (1)$$

90 We finally remark that slightly better bounds can be obtained in the case  $k = 4$ , using the  
91 bounds for biplanar crossing numbers by Czabarka, Sýkora, Székely, and Vrt'ó [6, 7].

## 92 1.4 $k$ -page drawings of $K_{m,n}$ for $k \geq 3$ : upper bounds

93 We found no references involving upper bounds of  $\nu_k(K_{m,n})$  in the literature. We note that  
94 since not every  $\lceil k/2 \rceil$ -planar drawing can be adapted to a  $k$ -page drawing, upper bounds  
95 for  $\lceil k/2 \rceil$ -planar crossing numbers do not yield upper bounds for  $k$ -page crossing numbers,  
96 and so the results on  $(k/2)$ -planar drawings of  $K_{m,n}$  in [22] cannot be used to derive upper  
97 bounds for  $\nu_k(K_{m,n})$ .

98 Below (cf. Theorem 6) we shall give general upper bounds for  $\nu_k(K_{m,n})$ . We derive these  
99 bounds using a natural construction, described in Section 7.

## 100 2 Main results

101 In this section we state the main new results in this paper, and briefly discuss the strategies  
102 of their proofs.

## 2.1 Exact pagenumbers

We have calculated the exact pagenumbers of several complete bipartite graphs:

**Theorem 3.** *For each  $k \in \{2, 3, 4, 5, 6\}$ , the pagenumber of  $K_{k+1, \lfloor (k+1)^2/4 \rfloor + 1}$  is  $k + 1$ .*

The proof of this statement is computer-aided, and is based on the formulation of  $\nu_k(K_{m,n})$  as a vertex coloring problem on an associated graph. This is presented in Section 3.

By the clever construction by Enomoto, Nakamigawa, and Ota [12],  $K_{k+1, \lfloor (k+1)^2/4 \rfloor}$  can be embedded into  $k$  pages, and so Theorem 3 implies (for  $k \in \{2, 3, 4, 5, 6\}$ ) that  $\lfloor \frac{(k+1)^2}{4} \rfloor + 1$  is the smallest value of  $n$  such that  $K_{k+1,n}$  does *not* embed in  $k$  pages. The case  $k = 2$  follows immediately from the nonplanarity of  $K_{3,3}$ ; we have included this value in the statement for completeness.

## 2.2 The $k$ -page crossing number of $K_{k+1, n}$ : exact results and bounds

Independently of the intrinsic value of learning some exact pagenumbers, the importance of Theorem 3 is that we need these results in order to establish the following general result. We emphasize that we follow the convention that  $\binom{a}{b} = 0$  whenever  $a < b$ .

**Theorem 4.** *Let  $k \in \{2, 3, 4, 5, 6\}$ , and let  $n$  be any positive integer. Define  $\ell := \lfloor \frac{(k+1)^2}{4} \rfloor$  and  $q := n \bmod \lfloor \frac{(k+1)^2}{4} \rfloor$ . Then*

$$\nu_k(K_{k+1,n}) = q \cdot \binom{\frac{n-q}{\ell} + 1}{2} + (\ell - q) \cdot \binom{\frac{n-q}{\ell}}{2}.$$

In this statement we have included the case  $k = 2$  again for completeness, as it asserts the well-known result that the 2-page crossing number of  $K_{3,n}$  equals  $Z(3, n) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ .

Although our techniques do not yield the exact value of  $\nu_k(K_{k+1,n})$  for other values of  $k$ , they give lower and upper bounds that imply sharp asymptotic estimates:

**Theorem 5.** *Let  $k, n$  be positive integers. Then*

$$2n^2 \left( \frac{1}{k^2 + 2000k^{7/4}} \right) - n < \nu_k(K_{k+1,n}) \leq \frac{2n^2}{k^2} + \frac{n}{2}.$$

Thus

$$\lim_{k \rightarrow \infty} \left( \lim_{n \rightarrow \infty} \frac{\nu_k(K_{k+1,n})}{2n^2/k^2} \right) = 1.$$

To grasp how this result relates to the bound in Theorem 2, let us note that the corresponding estimate (lower bound) from Theorem 2 is  $\lim_{k \rightarrow \infty} (\lim_{n \rightarrow \infty} \nu_k(K_{k+1,n}) / (2n^2/k^2)) \geq 1/4$ . Theorem 5 gives the exact asymptotic value of this quotient.

128 In a nutshell, the strategy to prove the lower bounds in Theorems 4 and 5 is to establish  
 129 lower bounds for  $\nu_k(K_{k+1,n})$  obtained under the assumption that  $\nu_k(K_{k+1,s+1})$  cannot be  
 130  $k$ -page embedded (for some integer  $s := s(k)$ ). These results put the burden of the proof of  
 131 the lower bounds in Theorems 4 and 5 in finding good estimates of  $s(k)$ . For  $k = 3, 4, 5, 6$   
 132 (Theorem 4), these come from Theorem 3, whereas for  $k > 6$  (Theorem 5) these are obtained  
 133 from [12, Theorem 5], which gives general estimates for such integers  $s(k)$ . The lower bounds  
 134 for  $\nu_k(K_{k+1,n})$  needed for both Theorems 4 and 5 are established in Section 4.

135 In Section 5 we prove the upper bounds on  $\nu_k(K_{k+1,n})$  claimed in Theorems 4 and 5. To  
 136 obtain these bounds, first we find a particular kind of  $k$ -page embeddings of  $K_{k+1, \lfloor (k+1)^2/4 \rfloor}$ ,  
 137 which we call *balanced* embeddings. These embeddings are inspired by, although not equal  
 138 to, the embeddings described by Enomoto et al. in [12]. We finally use these embeddings  
 139 to construct drawings of  $\nu_k(K_{k+1,n})$  with the required number of crossings.

140 Using the lower and upper bounds derived in Sections 4 and 5, respectively, Theorems 4  
 141 and 5 follow easily; their proofs are given in Section 6.

## 142 2.3 General upper bounds for $\nu_k(K_{m,n})$

143 As we mentioned above, we found no general upper bounds for  $\nu_k(K_{m,n})$  in the literature.  
 144 We came across a rather natural way of drawing  $K_{m,n}$  in  $k$  pages, that yields the general  
 145 upper bound given in the following statement.

146 **Theorem 6.** *Let  $k, m, n$  be nonnegative integers. Let  $r := m \bmod k$  and  $s := n \bmod k$ .  
 147 Then*

$$\nu_k(K_{m,n}) \leq \frac{(m-r)(n-s)}{4k^2} (m-k+r)(n-k+s) \leq \frac{1}{k^2} \binom{m}{2} \binom{n}{2}.$$

148 The proof of this statement is given in Section 7.

## 149 2.4 $k$ -page vs. $(k/2)$ -planar crossing numbers

150 As we have already observed, for every even integer  $k$ , every  $k$ -page drawing can be regarded  
 151 as a  $(k/2)$ -planar drawing. Thus, for every graph  $G$ ,  $\text{cr}_{k/2}(G) \leq \nu_k(G)$ .

152 Since there is (at least in principle) considerable more freedom in a  $(k/2)$ -planar drawing  
 153 than in a  $k$ -page drawing, it is natural to ask whether or not this additional freedom can  
 154 be translated into a substantial saving in the number of crossings. For small values of  $m$  or  
 155  $n$ , the answer is yes. Indeed, Beineke [1] described how to draw  $K_{k+1,k(k-1)}$  in  $k/2$  planes  
 156 without crossings, but by Proposition 15,  $K_{k+1,k^2/4+500k^{7/4}}$  cannot be  $k$ -page embedded;  
 157 thus the  $k/2$ -planar crossing number of  $K_{k+1,k(k-1)}$  is 0, whereas its  $k$ -page crossing number  
 158 can be arbitrarily large. Thus it makes sense to ask about the asymptotic behaviour when  
 159  $k, m$ , and  $n$  all go to infinity. Letting  $\gamma(k) := \lim_{m,n \rightarrow \infty} \text{cr}_{k/2}(K_{m,n})/\nu_k(K_{m,n})$ , we focus on  
 160 the question: is  $\lim_{k \rightarrow \infty} \gamma(k) = 1$ ?

161 Since we do not know (even asymptotically) the  $(k/2)$ -planar or the  $k$ -page crossing number  
 162 of  $K_{m,n}$ , we can only investigate this question in the light of the current best bounds  
 163 available.

164 In Section 8 we present a discussion around this question. We conclude that if the  $(k/2)$ -  
 165 planar and the  $k$ -page crossing numbers (asymptotically) agree with the current best upper  
 166 bounds, then indeed the limit above equals 1. We also observe that this is not the case for  
 167 complete graphs: the currently best known  $(k/2)$ -planar drawings of  $K_n$  are substantially  
 168 better (even asymptotically) than the currently best known  $k$ -page drawings of  $K_n$ .

### 169 3 Exact pagenumbers: proof of Theorem 3

170 We start by observing that for every integer  $n$ , the graph  $K_{k+1,n}$  can be embedded in  
 171  $k + 1$  pages, and so the pagenumber  $p(K_{k+1, \lfloor (k+1)^2/4 \rfloor + 1})$  of  $K_{k+1, \lfloor (k+1)^2/4 \rfloor + 1}$  is at most  
 172  $k + 1$ . Thus we need to show the reverse inequality  $p(K_{k+1, \lfloor (k+1)^2/4 \rfloor + 1}) \geq k + 1$ , for  
 173 every  $k \in \{3, 4, 5, 6\}$ . It clearly suffices to show that  $\nu_k(K_{k+1, \lfloor (k+1)^2/4 \rfloor + 1}) > 0$ , for every  
 174  $k \in \{3, 4, 5, 6\}$ .

175 These inequalities are equivalent to  $k$ -colorability of certain auxiliary graphs. To this end,  
 176 we define an auxiliary graph  $G_D(K_{m,n})$  associated with a 1-page (circular) drawing  $D$  of  
 177  $K_{m,n}$  as follows. The vertices of  $G_D(K_{m,n})$  are the edges of  $K_{m,n}$ , and two vertices are  
 178 adjacent if the corresponding edges cross in the drawing  $D$ .

179 We immediately have the following result, that is essentially due to Buchheim and Zheng [4].

180 **Lemma 7** (cf. Buchheim-Zheng [4]). *One has  $\nu_k(K_{m,n}) > 0$  if and only if the chromatic*  
 181 *number of  $G_D(K_{m,n})$  is greater than  $k$  for all circular drawings  $D$  of  $K_{m,n}$ .*

182 As a consequence we may decide if  $\nu_k(K_{m,n}) > 0$  by considering all possible circular draw-  
 183 ings  $D$  of  $K_{m,n}$ , and computing the chromatic numbers of the associated auxiliary graphs  
 184  $G_D(K_{m,n})$ . The number of distinct circular drawings of  $K_{m,n}$  may be computed using  
 185 the classical *orbit counting lemma*, often attributed to Burnside, although it was certainly  
 186 already known to Frobenius.

**Lemma 8** (Orbit counting lemma). *Let a finite group  $\mathcal{G}$  act on a finite set  $\Omega$ . Denote by*  
 *$\Omega^g$ , for  $g \in \mathcal{G}$ , the set of elements of  $\Omega$  fixed by  $g$ . Then the number  $N$  of orbits of  $\mathcal{G}$  on  $\Omega$*   
*is the average, over  $\mathcal{G}$ , of  $|\Omega^g|$ , i.e.*

$$N = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} |\Omega^g|.$$

187 We will apply this lemma by considering that a circular drawing of  $K_{m,n}$  is uniquely de-  
 188 termined by the ordering of the  $m$  blue and  $n$  red vertices on a circle. We therefore define  
 189 the finite set  $\Omega$  as the set of all  $\binom{m+n}{n}$  such orderings. Now consider the usual action of  
 190 the dihedral group  $\mathcal{G} := D_{m+n}$  on the set  $\Omega$ . For our purposes two orderings are the same,

191 i.e. correspond to the same circular drawing of  $K_{m,n}$ , if they belong to the same orbit of  $\mathcal{G}$ .  
 192 We therefore only need to count the number of orbits by using the last lemma. The final  
 193 result is as follows. (We omit the details of the counting argument, as it is a straightforward  
 194 exercise in combinatorics.)

195 **Lemma 9.** *Let  $m$  and  $n$  be positive integers and denote  $d = \gcd(m, n)$ . The number of*  
 196 *distinct circular drawings of  $K_{m,n}$  equals:*

$$\frac{1}{2(m+n)} \begin{cases} \frac{m+n}{2} \left( \binom{\frac{m+n}{2}}{n/2} + \binom{\frac{m+n-2}{2}}{m/2} + \binom{\frac{m+n-2}{2}}{n/2} \right) + \sum_{k=0}^{d-1} \binom{\frac{m+n}{o(k)}}{\frac{m}{o(k)}} & (m, n \text{ even}), \\ (m+n) \binom{\frac{m+n-1}{2}}{n/2} + \sum_{k=0}^{d-1} \binom{\frac{m+n}{o(k)}}{\frac{m}{o(k)}} & (m \text{ odd}, n \text{ even}), \\ (m+n) \binom{\frac{m+n-2}{2}}{(m-1)/2} + \sum_{k=0}^{d-1} \binom{\frac{m+n}{o(k)}}{\frac{m}{o(k)}} & (m, n \text{ odd}), \end{cases}$$

197 where  $o(k)$  is the minimal number between 1 and  $d$  such that  $k \cdot o(k) \equiv 0 \pmod{d}$ . In other  
 198 words,  $o(k)$  is the order of the subgroup generated by  $k$  in the additive group of integers  
 199 mod  $d$ .

200 In what follows we will present computer-assisted proofs that the chromatic number of  
 201  $G_D(K_{m,n})$  is greater than  $k$ , for specific integers  $k, m, n$ . We do not need to compute the  
 202 chromatic number exactly if we can prove that it is lower bounded by a value strictly greater  
 203 than  $k$ . A suitable lower bound for our purposes is the Lovász  $\vartheta$ -number.

204 **Lemma 10** (Lovász [17]). *Given a graph  $G = (V, E)$  and the value*

$$\vartheta(G) := \max_{X \succeq 0} \left\{ \sum_{i,j \in V} X_{ij} \mid X_{ij} = 0 \text{ if } (i, j) \in E, \text{ trace}(X) = 1, X \in \mathbb{R}^{V \times V} \right\},$$

205 one has

$$\omega(\bar{G}) \leq \vartheta(G) \leq \chi(\bar{G}),$$

206 where  $\omega(\bar{G})$  and  $\chi(\bar{G})$  are the clique and chromatic numbers of the complement  $\bar{G}$  of  $G$ ,  
 207 respectively.

208 The  $\vartheta(G)$ -number may be computed for a given graph  $G$  by using semidefinite programming  
 209 software. For our computation we used the software DSDP [3].

210 **Corollary 11.** *If, for given positive integers  $m, n$  and  $k$ ,  $\vartheta(\overline{G_D(K_{m,n})}) > k$  for all circular*  
 211 *drawings  $D$  of  $K_{m,n}$ , then  $\nu_k(K_{m,n}) > 0$ .*

212 If, for a given circular drawing  $D$ , we find that  $\vartheta(\overline{G_D(K_{m,n})}) = k$ , then we compute the  
 213 chromatic number of  $G_D(K_{m,n})$  exactly, by using satisfiability or integer programming  
 214 software. For our computation we used the satisfiability solver Akmaxsat [15], and for



the integer programming formulation the solver XPRESS-MP [16]. The formulation of the chromatic number as the solution of a maximum satisfiability problem is described in [10, §3.3]. The integer programming formulation we used is the following.

For given  $G = (V, E)$  with adjacency matrix  $A$ , and set of colors  $C = \{1, \dots, k\}$ , define the binary variables

$$x_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is assigned color } j, \\ 0 & \text{else,} \end{cases} \quad (i \in V, j \in C),$$

and consider the integer programming feasibility problem:

$$\text{Find an } x \in \{0, 1\}^{V \times C} \text{ such that } \sum_{j \in C} x_{ij} = 1 \ \forall i \in V, \sum_{i \in V} A_{pi} x_{ij} \leq |E|(1 - x_{pj}) \ \forall p \in V, j \in C. \quad (2)$$

**Lemma 12.** *A given graph  $G = (V, E)$  is  $k$ -colorable if and only if the integer program (2) has a solution.*

We may therefore solve (2) with  $G = G_D(K_{m,n})$ , for each circular drawing  $D$  of  $K_{m,n}$ , to decide if  $\nu_k(K_{m,n}) > 0$ .

Finally we describe the results we obtained by using the computational framework described in this section.

**Case  $k = 3$ : proof of  $\nu_3(K_{4,5}) > 0$ .**

By Lemma 9, there are 10 distinct circular drawings  $D$  of  $K_{4,5}$ . For each  $D$  we showed numerically that  $\vartheta(\overline{G_D(K_{4,5})}) > 3$ . The required result now follows from Corollary 11.

**Case  $k = 4$ : proof of  $\nu_4(K_{5,7}) > 0$ .**

By Lemma 9, there are 38 distinct circular drawings  $D$  of  $K_{5,7}$ . For all but one  $D$  we showed numerically that  $\vartheta(\overline{G_D(K_{5,7})}) > 4$ . The remaining case was settled by showing  $\chi(G_D(K_{5,7})) > 4$  using the satisfiability reformulation. The required result now follows from Corollary 11 and Lemma 7.

**Case  $k = 5$ : proof of  $\nu_5(K_{6,10}) > 0$ .**

By Lemma 9, there are 210 distinct circular drawings  $D$  of  $K_{6,10}$ . For all but one  $D$  we showed numerically that  $\vartheta(\overline{G_D(K_{6,10})}) > 5$ . The remaining case was settled by showing  $\chi(G_D(K_{6,10})) > 5$  using the satisfiability reformulation. The required result now follows from Corollary 11 and Lemma 7.

240 **Case  $k = 6$ : proof of  $\nu_6(K_{7,13}) > 0$ .**

241 By Lemma 9, there are 1980 distinct circular drawings  $D$  of  $K_{7,13}$ . For all but one  $D$  we  
 242 showed numerically that  $\vartheta(\overline{G_D(K_{7,13})}) > 6$ . The remaining case was settled by showing  
 243  $\chi(G_D(K_{7,13})) > 4$  using the integer programming reformulation (2). The required result  
 244 now follows from Corollary 11, Lemma 12, and Lemma 7.

## 245 4 $k$ -page crossing numbers of $K_{k+1,n}$ : lower bounds

246 Our aim in this section is to establish lower bounds for  $\nu_k(K_{k+1,n})$ . Our strategy is as follows.  
 247 First we find (Proposition 13) a lower bound under the assumption that  $K_{k+1,s+1}$  cannot be  
 248  $k$ -page embedded (for some integer  $s := s(k)$ ). Then we find values of  $s$  such that  $K_{k+1,s+1}$   
 249 cannot be  $k$ -page embedded; these are given in Propositions 14 (for  $k \in \{2, 3, 4, 5, 6\}$ ) and 15  
 250 (for every  $k$ ). We then put these results together and establish the lower bounds required  
 251 in Theorem 4 (see Lemma 16) and in Theorem 5 (see Lemma 17).

252 **Proposition 13.** *Suppose that  $K_{k+1,s+1}$  cannot be  $k$ -page embedded. Let  $n$  be a positive*  
 253 *integer, and define  $q := n \bmod s$ . Then*

$$\nu_k(K_{k+1,n}) \geq q \cdot \binom{\frac{n-q}{s} + 1}{2} + (s - q) \cdot \binom{\frac{n-q}{s}}{2}.$$

254 *Proof.* It is readily verified that if  $n \leq s$  then the right hand side of the inequality in the  
 255 proposition equals 0, and so in this case the inequality trivially holds. Thus we may assume  
 256 that  $n \geq s + 1$ .

257 Let  $D$  be a  $k$ -page drawing of  $K_{k+1,n}$ . Construct an auxiliary graph  $G$  as follows. Let  $V(G)$   
 258 be the set of  $n$  degree- $(k+1)$  vertices in  $K_{k+1,n}$ , and join two vertices  $u, v$  in  $G$  by an edge  
 259 if there are edges  $e_u, e_v$  incident with  $u$  and  $v$  (respectively) that cross each other in  $D$ .

260 Since  $K_{k+1,s+1}$  cannot be embedded in  $k$  pages, it follows that  $G$  has no independent set of  
 261 size  $s + 1$ . Equivalently, the complement graph  $\overline{G}$  of  $G$  has no clique of size  $s + 1$ . Turán's  
 262 theorem asserts that  $\overline{G}$  cannot have more edges than the Turán graph  $T(n, s)$ , and so  $G$   
 263 has at least as many edges as the complement  $\overline{T}(n, s)$  of  $T(n, s)$ . We recall that  $\overline{T}(n, s)$  is  
 264 formed by the disjoint union of  $s$  cliques,  $q$  of them with  $(n - q)/s + 1$  vertices, and  $s - q$   
 265 of them with  $(n - q)/s$  vertices. Thus

$$|E(G)| \geq q \cdot \binom{\frac{n-q}{s} + 1}{2} + (s - q) \cdot \binom{\frac{n-q}{s}}{2}.$$

266 Since clearly the number of crossings in  $D$  is at least  $|E(G)|$ , and  $D$  is an arbitrary  $k$ -page  
 267 drawing of  $K_{k+1,n}$ , the result follows.  $\square$

268 **Proposition 14.** *For each  $k \in \{2, 3, 4, 5, 6\}$ ,  $K_{k+1, \lfloor \frac{(k+1)^2}{4} \rfloor + 1}$  cannot be  $k$ -page embedded.*

269 *Proof.* This is an immediate consequence of Theorem 3.  $\square$

270 **Proposition 15.** *For each positive integer  $k$ , the graph  $K_{k+1, k^2/4+500k^{7/4}}$  cannot be  $k$ -page*  
 271 *embedded.*

272 *Proof.* Define  $g(n) := \min\{m \mid \text{the pagenumber of } K_{m,n} \text{ is } n\}$ . Enomoto et al. proved that  
 273  $g(n) = n^2/4 + O(n^{7/4})$  ([12, Theorem 5]). Our aim is simply to get an explicit estimate  
 274 of the  $O(n^{7/4})$  term (without making any substantial effort to optimize the coefficient of  
 275  $n^{7/4}$ ).

276 In the proof of [12, Theorem 5], Enomoto et al. gave upper bounds for three quantities  
 277  $m_1, m_2, m_3$ , and proved that  $g(n) \leq m_1 + m_2 + m_3$ . They showed  $m_1 \leq n^{3/4}(n-r)$  (for  
 278 certain  $r \leq n$ ),  $m_2 \leq (n^{1/4} + 1)(2n^{1/4} + 2)(n-1)$ , and  $m_3 \leq (n^{1/4} + 1)(n^{1/4} + 2)(2n^{1/4} +$   
 279  $3)(n-1) + r'(n-r')$  (for certain  $r' \leq r$ ).

280 Noting that  $r' \leq r$ , the inequality  $m_1 \leq n^{3/4}(n-r)$  gives  $m_1 \leq n^{3/4}(n-r')$ . Elementary  
 281 manipulations give  $m_2 < 2n(n^{1/4} + 1)^2 = 2n(n^{1/2} + 2n^{1/4} + 1) \leq 2n(4n^{1/2}) = 8n^{3/2} < 8n^{7/4}$ ,  
 282 and  $m_3 < n(2n^{1/4} + 3)^3 + r'(n-r') < n(5n^{1/4})^3 + r'(n-r') = 125n^{7/4} + r'(n-r')$ . Thus  
 283 we obtain  $g(n) \leq m_1 + m_2 + m_3 < 133n^{7/4} + (n^{3/4} + r')(n-r')$ . An elementary calculus  
 284 argument shows that  $(n^{3/4} + r')(n-r')$  is maximized when  $r' = (n - n^{3/4})/2$ , in which case  
 285  $(n^{3/4} + r')(n-r') = (1/4)(n^{3/4} + n)^2 = (1/4)(n^{3/2} + 2n^{7/4} + n^2) < (1/4)(n^2 + 3n^{7/4})$ .

286 Thus we get  $g(n) < 133n^{7/4} + n^2/4 + (3/4)n^{7/4} < n^2/4 + 134n^{7/4}$ , and so  $g(k+1) <$   
 287  $(k+1)^2/4 + 134(k+1)^{7/4} < k^2/4 + k/2 + 1/4 + 134(2k)^{7/4} < k^2/4 + 136(2k)^{7/4} = k^2/4 +$   
 288  $136 \cdot 2^{7/4} \cdot k^{7/4} < k^2/4 + 500k^{7/4}$ . This means, from the definition of  $g$ , that  $K_{k+1, k^2/4+500k^{7/4}}$   
 289 cannot be  $k$ -page embedded.  $\square$

290 **Lemma 16.** *For each  $k \in \{2, 3, 4, 5, 6\}$ , and every integer  $n$ ,*

$$\nu_k(K_{k+1,n}) \geq q \cdot \binom{\frac{n-q}{\ell} + 1}{2} + (\ell - q) \cdot \binom{\frac{n-q}{\ell}}{2},$$

291 *where  $\ell := \lfloor (k+1)^2/4 \rfloor$  and  $q := n \bmod \lfloor (k+1)^2/4 \rfloor$ .*

292 *Proof.* It follows immediately from Propositions 13 and 14.  $\square$

293 **Lemma 17.** *For all positive integers  $k$  and  $n$ ,*

$$\nu_k(K_{k+1,n}) > 2n^2 \left( \frac{1}{k^2 + 2000k^{7/4}} \right) - n.$$

*Proof.* By Proposition 15 it follows that  $K_{k+1, k^2/4+500k^{7/4}}$  cannot be  $k$ -page embedded.  
 Thus, if we let  $s := k^2/4 + 500k^{7/4} - 1$  and  $q := n \bmod s$ , it follows from Proposition 13 that  
 $\nu_k(K_{k+1,n}) \geq q \cdot \binom{\frac{n-q}{s} + 1}{2} + (s-q) \cdot \binom{\frac{n-q}{s}}{2} \geq s \cdot \binom{\frac{n-q}{s} + 1}{2} > (n-q)^2/2s$ . Thus we have

$$\nu_k(K_{k+1,n}) > \frac{(n-q)^2}{2s} > \frac{(n-s)^2}{2s} > \frac{n^2}{2s} - n > 2n^2 \left( \frac{1}{k^2 + 2000k^{7/4}} \right) - n. \quad \square$$

## 294 5 $k$ -page crossing numbers of $K_{k+1,n}$ : upper bounds

295 In this section we derive an upper bound for the  $k$ -page crossing number of  $K_{k+1,n}$ , which  
 296 will yield the upper bounds claimed in both Theorems 4 and 5.

297 To obtain this bound we proceed as follows. First, we show in Proposition 18 that if for  
 298 some  $s$  the graph  $K_{k+1,s}$  admits a certain kind of  $k$ -page embedding (what we call a *balanced*  
 299 embedding), then this embedding can be used to construct drawings of  $\nu_k(K_{k+1,n})$  with a  
 300 certain number of crossings. Then we prove, in Proposition 19, that  $K_{k+1,\lfloor (k+1)^2/4 \rfloor}$  admits  
 301 a balanced  $k$ -page embedding for every  $k$ . These results are then put together to obtain  
 302 the required upper bound, given in Lemma 20.

### 303 5.1 Extending balanced $k$ -page embeddings to $k$ -page drawings

304 We consider  $k$ -page embeddings of  $K_{k+1,s}$ , for some integers  $k$  and  $s$ . To help comprehension,  
 305 color the  $k + 1$  degree- $s$  vertices *black*, and the  $s$  degree- $(k + 1)$  vertices *white*. Given such  
 306 an embedding, a white vertex  $v$ , and a page, the *load* of  $v$  in this page is the number of  
 307 edges incident with  $v$  that lie on the given page.

308 The pigeon-hole principle shows that in an  $k$ -page embedding of  $K_{k+1,s}$ , for each white  
 309 vertex  $v$  there must exist a page with load at least 2. A  $k$ -page embedding of  $K_{k+1,s}$  is  
 310 *balanced* if for each white vertex  $v$ , there exist  $k - 1$  pages in which the load of  $v$  is 1 (and  
 311 so the load of  $v$  in the other page is necessarily 2).

312 **Proposition 18.** *Suppose that  $K_{k+1,s}$  admits a balanced  $k$ -page embedding. Let  $n \geq s$ , and*  
 313 *define  $q := n \bmod s$ . Then*

$$\nu_k(K_{k+1,n}) \leq q \cdot \binom{\frac{n-q}{s} + 1}{2} + (s - q) \cdot \binom{\frac{n-q}{s}}{2}.$$

314 *Proof.* Let  $\Psi$  be a balanced  $k$ -page embedding of  $K_{k+1,s}$ , presented in the circular model.  
 315 To construct from  $\Psi$  a  $k$ -page drawing of  $K_{k+1,n}$ , we first “blow up” each white point as  
 316 follows.

317 Let  $t \geq 1$  be an integer. Consider a white point  $r$  in the circle, and let  $N_r$  be a small  
 318 neighborhood of  $r$ , such that no point (black or white) other than  $r$  is in  $N_r$ . Now place  
 319  $t - 1$  additional white points on the circle, all contained in  $N_r$ , and let each new white point  
 320 be joined to a black point  $b$  (in a given page) if and only if  $r$  is joined to  $b$  in that page. We  
 321 say that the white point  $r$  has been *converted into a  $t$ -cluster*.

322 To construct a  $k$ -page drawing of  $K_{k+1,n}$ , we start by choosing (any)  $q$  white points, and  
 323 then convert each of these  $q$  white points into an  $((n - q)/s + 1)$ -cluster. Finally, convert  
 324 each of the remaining  $s - q$  white points into an  $((n - q)/s)$ -cluster. The result is evidently  
 325 an  $k$ -page drawing  $D$  of  $K_{k+1,n}$ .

326 We finally count the number of crossings in  $D$ . Consider the  $t$ -cluster  $C_r$  obtained from  
 327 some white point  $r$  (thus,  $t$  is either  $(n - q)/s$  or  $(n - q)/s + 1$ ), and consider any page  $\pi_i$ .  
 328 It is clear that if the load of  $r$  in  $\pi_i$  is 1, then no edge incident with a vertex in  $C_r$  is crossed  
 329 in  $\pi_i$ . On the other hand, if the load of  $r$  in  $\pi_i$  is 2, then it is immediately checked that  
 330 the number of crossings involving edges incident with vertices in  $C_r$  is exactly  $\binom{t}{2}$ . Now  
 331 the load of  $r$  is 2 in exactly one page (since  $\Psi$  is balanced), and so it follows that the total  
 332 number of crossings in  $D$  involving edges incident with vertices in  $C_r$  is  $\binom{t}{2}$ . Since to obtain  
 333  $D$ ,  $q$  white points were converted into  $((n - q)/s + 1)$ -clusters, and  $s - q$  white points were  
 334 converted into  $((n - q)/s)$ -clusters, it follows that the number of crossings in  $D$  is exactly

$$q \cdot \binom{\frac{n-q}{s} + 1}{2} + (s - q) \cdot \binom{\frac{n-q}{s}}{2}. \quad \square$$

## 335 5.2 Constructing balanced $k$ -page embeddings

336 Enomoto, Nakamigawa, and Ota [12] gave a clever general construction to embed  $K_{m,n}$   
 337 in  $s$  pages for (infinitely) many values of  $m, n$ , and  $s$ . In particular, their construction  
 338 yields  $k$ -page embeddings of  $K_{k+1, \lfloor (k+1)^2/4 \rfloor}$ . However, the embeddings obtained from their  
 339 technique are not balanced (see Figure 2). We have adapted their construction to establish  
 340 the following.

341 **Proposition 19.** *For each positive integer  $k$ , the graph  $K_{k+1, \lfloor (k+1)^2/4 \rfloor}$  admits a balanced*  
 342  *$k$ -page embedding.*

343 *Proof.* We show that for each pair of positive integers  $s, t$  such that  $t$  is either  $s$  or  $s + 1$ , the  
 344 graph  $K_{s+t, st}$  admits a balanced  $(s + t - 1)$ -page embedding. The proposition then follows:  
 345 given  $k$ , if we set  $s := \lfloor (k + 1)/2 \rfloor$  and  $t := \lceil (k + 1)/2 \rceil$ , then  $t \in \{s, s + 1\}$ , and clearly  
 346  $k + 1 = s + t$  (and so  $k = s + t - 1$ ) and  $\lfloor \frac{(k+1)^2}{4} \rfloor = st$ .

347 To help comprehension, we color the  $s + t$  degree- $st$  vertices *black*, and we color the  $st$   
 348 degree- $(s + t)$  vertices *white*. We describe the required embedding using the circular model.  
 349 Thus, we start with  $s + t - 1$  pairwise disjoint copies of a circle; these copies are the pages  
 350  $0, 1, \dots, s + t - 2$ . In the boundary of each copy we place the  $s + t + st$  vertices, so that  
 351 the vertices are placed in an identical manner in all  $s + t - 1$  copies. Each edge will be  
 352 drawn in the interior of the circle of exactly one page, using the straight segment joining  
 353 the corresponding vertices.

354 We now describe how we arrange the white and the black points on the circle boundary. We  
 355 use the black-and-white arrangement proposed by Enomoto et al. [12]. We refer the reader  
 356 to Figure 3. First we place the  $s + t$  black points  $b_0, b_1, \dots, b_{s+t-1}$  in the circle boundary,  
 357 in this clockwise cyclic order. Now for each  $i \in \{0, 1, \dots, t - 1\}$ , we insert between the  
 358 vertices  $b_{s+i}$  and  $b_{s+i+1}$  a collection  $w_{is}, w_{is+1}, \dots, w_{is+s-1}$  of white vertices, also listed in  
 359 the clockwise cyclic order in which they appear between  $b_{s+i}$  and  $b_{s+i+1}$  (operations on the  
 360 indices of the black vertices are modulo  $s + t$ ). For any  $i, j$  such that  $0 \leq i \leq j < st$ , we  
 361 let  $W[i : j]$  denote the set of white vertices  $\{w_i, w_{i+1}, \dots, w_j\}$ . For  $i = \{0, 1, \dots, t - 1\}$ ,

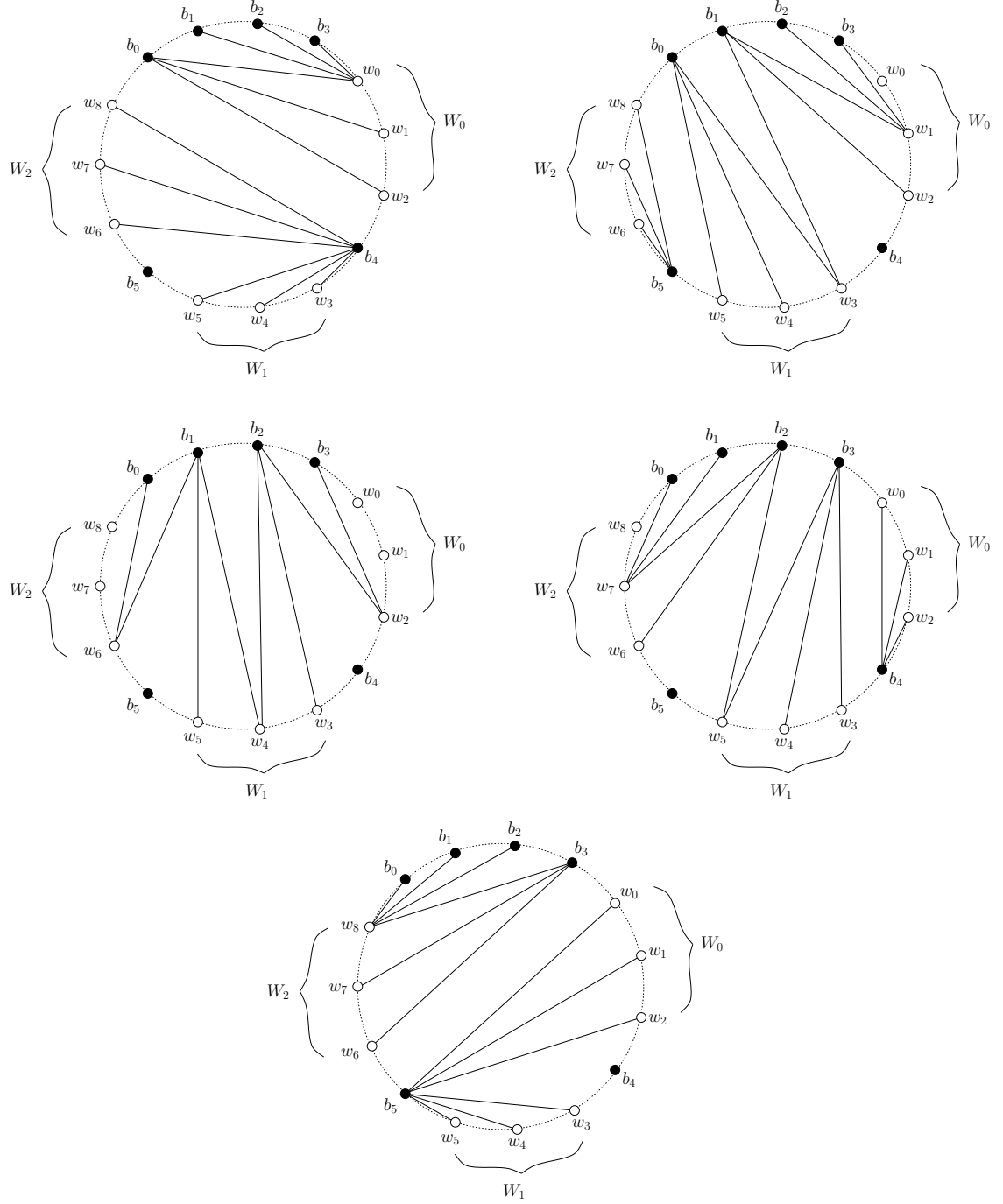


Figure 2: The 5-page embedding of  $K_{6,9}$  obtained from the general construction of Enomoto, Nakamigawa, and Ota. This embedding is not balanced (for instance, the white vertex  $w_0$  has degree 4 in Page 0).

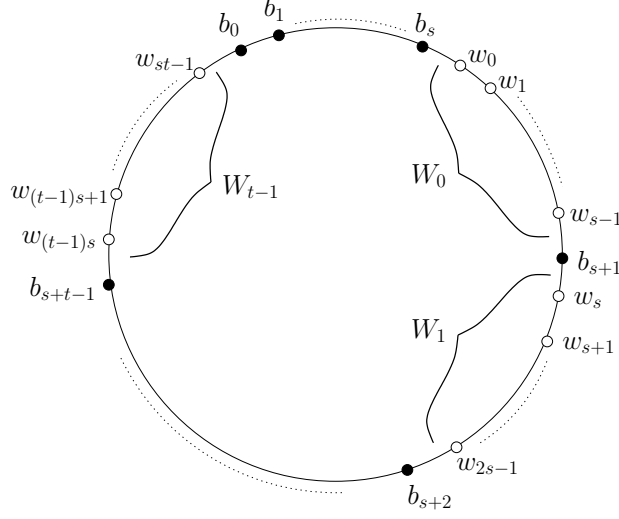


Figure 3: Layout of the vertices of  $K_{s+t, st}$ .

we call the set  $W[is : is + s - 1] = \{w_{is}, w_{is+1}, \dots, w_{is+s-1}\}$  a *white block*, and denote it by  $W_i$ . Thus the whole collection of white vertices  $w_0, w_1, \dots, w_{st-1}$  is partitioned into  $t$  blocks  $W_0, W_1, \dots, W_{t-1}$ , each of size  $s$ . Note that the black vertices  $b_0, b_1, \dots, b_s$  occur consecutively in the circle boundary (that is, no white vertex is between  $b_i$  and  $b_{i+1}$ , for  $i \in \{0, 1, \dots, s-1\}$ ). On the other hand, for  $i = s+1, s+2, \dots, s+t-1$ , the black vertex  $b_i$  occurs between two white vertices: loosely speaking,  $b_i$  is sandwiched between the white blocks  $W_{i-1}$  and  $W_i$  (operations on the indices of the white blocks are modulo  $t$ ).

Now we proceed to place the edges on the pages. We refer the reader to Figures 4 and 5 for illustrations of the edges distributions for the cases  $k = 5$  and  $6$ . We remark that: (i) operations on page numbers are modulo  $s+t-1$ ; (ii) operations on block indices are modulo  $t$ ; (iii) operations on the indices of black vertices are modulo  $s+t$ ; and (iv) operations on the indices of white vertices are modulo  $st$ .

For  $r = 0, 1, \dots, s-1$ , place the following edges in page  $r$ :

TYPE I For  $i = r+1, r+2, \dots, t$ , the edges joining  $b_{s+i}$  to all the vertices in the white block  $W_{t+r-i}$  (note that  $b_{s+t} = b_0$ ).

TYPE II For  $0 < i < r+1$ , the edges joining  $b_i$  to all the vertices in  $W[rs - i(s-1) : rs - (i-1)(s-1)]$ .

TYPE III The edges joining  $b_{r+1}$  to all the vertices in  $W[0 : r]$ .

For  $r = s, s+1, \dots, s+t-2$ , place the following edges in page  $r$ :

TYPE IV For  $i = 0, 1, \dots, r-s+1$ ,  $b_{s+i}$  to all the vertices in the white block  $W_{r-s-i+1}$ .

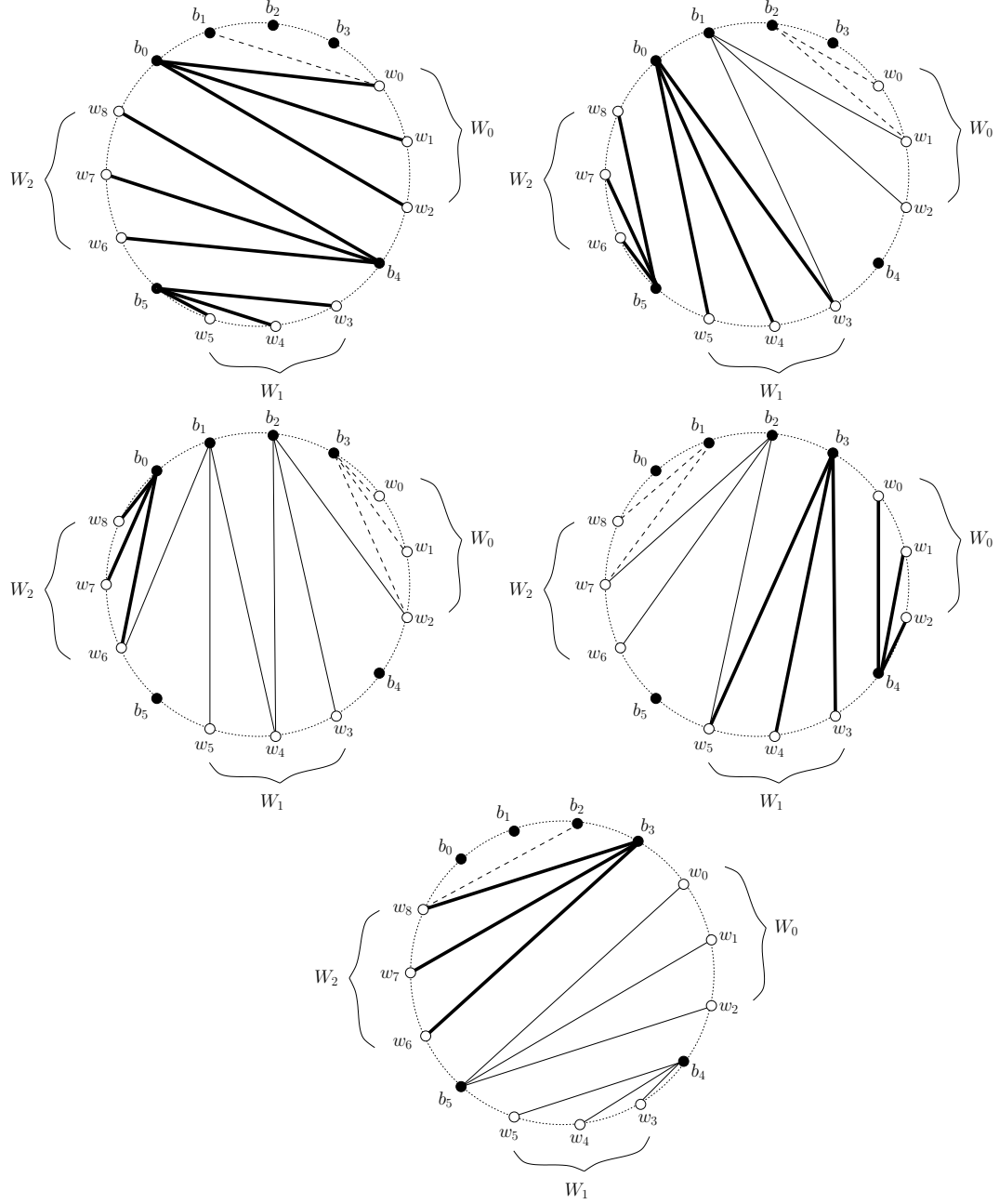


Figure 4: A balanced 5-page embedding of  $K_{6,9}$ . In this case  $k = 5$ , and so  $s = t = 3$ . Pages 0, 1, 2, 3 and 4 are the upper left, upper right, middle left, middle right, and lower circle, respectively. For Pages 0, 1, and 2, we have edges of Types I, II, and III, whereas for Pages 3 and 4, we have edges of Types IV, V, and VI. Edges of Types I and IV are drawn with thick segments; edges of Types II and V are drawn with thinner segments; and edges of Types III and VI are drawn with dashed segments.



382 TYPE V For  $0 < i < s - r + t - 1$ , the edges joining  $b_{s-i}$  to all the vertices in  $W[(i +$   
383  $r - s + 1)s - i : (i + r - s + 1)s - i + (s - 1)]$ .

384 TYPE VI The edges joining  $b_{r-t+1}$  to all the vertices in  $W[st - t + r - s + 1 : st - 1]$ .

385 It is a tedious but straightforward task to check that this yields an  $(s+t-1)$ -page embedding  
386 of  $K_{s+t, st}$ . Moreover, since every white vertex has load at least 1 in every page, it follows  
387 immediately that the embedding is balanced.  $\square$

### 388 5.3 The upper bound

389 **Lemma 20.** *For all positive integers  $k$  and  $n$ ,*

$$\nu_k(K_{k+1, n}) \leq q \cdot \binom{\frac{n-q}{\ell} + 1}{2} + (\ell - q) \cdot \binom{\frac{n-q}{\ell}}{2},$$

390 where  $\ell := \lfloor (k+1)^2/4 \rfloor$  and  $q := n \bmod \lfloor (k+1)^2/4 \rfloor$ .

391 *Proof.* It follows immediately by combining Propositions 18 and 19.  $\square$

## 392 6 Proofs of Theorems 4 and 5

393 We first observe that Theorem 4 follows immediately by combining Lemmas 16 and 20.

Now to prove Theorem 5, we let  $\ell := \lfloor (k+1)^2/4 \rfloor$  and  $q := n \bmod \lfloor (k+1)^2/4 \rfloor$ , and note that it follows from Lemma 20 that

$$\begin{aligned} \nu_k(K_{k+1, n}) &\leq q \cdot \binom{\frac{n-q}{\ell} + 1}{2} + (\ell - q) \cdot \binom{\frac{n-q}{\ell}}{2} \leq \ell \cdot \binom{\frac{n-q}{\ell} + 1}{2} = \frac{\ell}{2} \cdot \left( \frac{n-q}{\ell} + 1 \right) \left( \frac{n-q}{\ell} \right) \\ &= \frac{n-q}{2} \cdot \left( \frac{n-q}{\ell} + 1 \right) \leq \frac{n}{2} \cdot \left( \frac{n}{\ell} + 1 \right) = \frac{n^2}{2\ell} + \frac{n}{2} \leq \frac{n^2}{2(k^2/4)} + \frac{n}{2} = \frac{2n^2}{k^2} + \frac{n}{2}. \end{aligned}$$

394 Combining this with Lemma 17, we obtain

$$2n^2 \left( \frac{1}{k^2 + 2000k^{7/4}} \right) - n < \nu_k(K_{k+1, n}) \leq \frac{2n^2}{k^2} + \frac{n}{2},$$

395 proving Theorem 5.

## 396 7 A general upper bound for $\nu_k(K_{m, n})$ : proof of Theorem 6

397 We now describe a quite natural construction to draw  $K_{m, n}$  in  $k$  pages, for every  $k \geq 3$ .  
398 Actually, our construction also works for the case  $k = 2$ , and for this case the upper bounds  
399 obtained coincide with the best known upper bound for  $\nu_2(K_{m, n})$ .

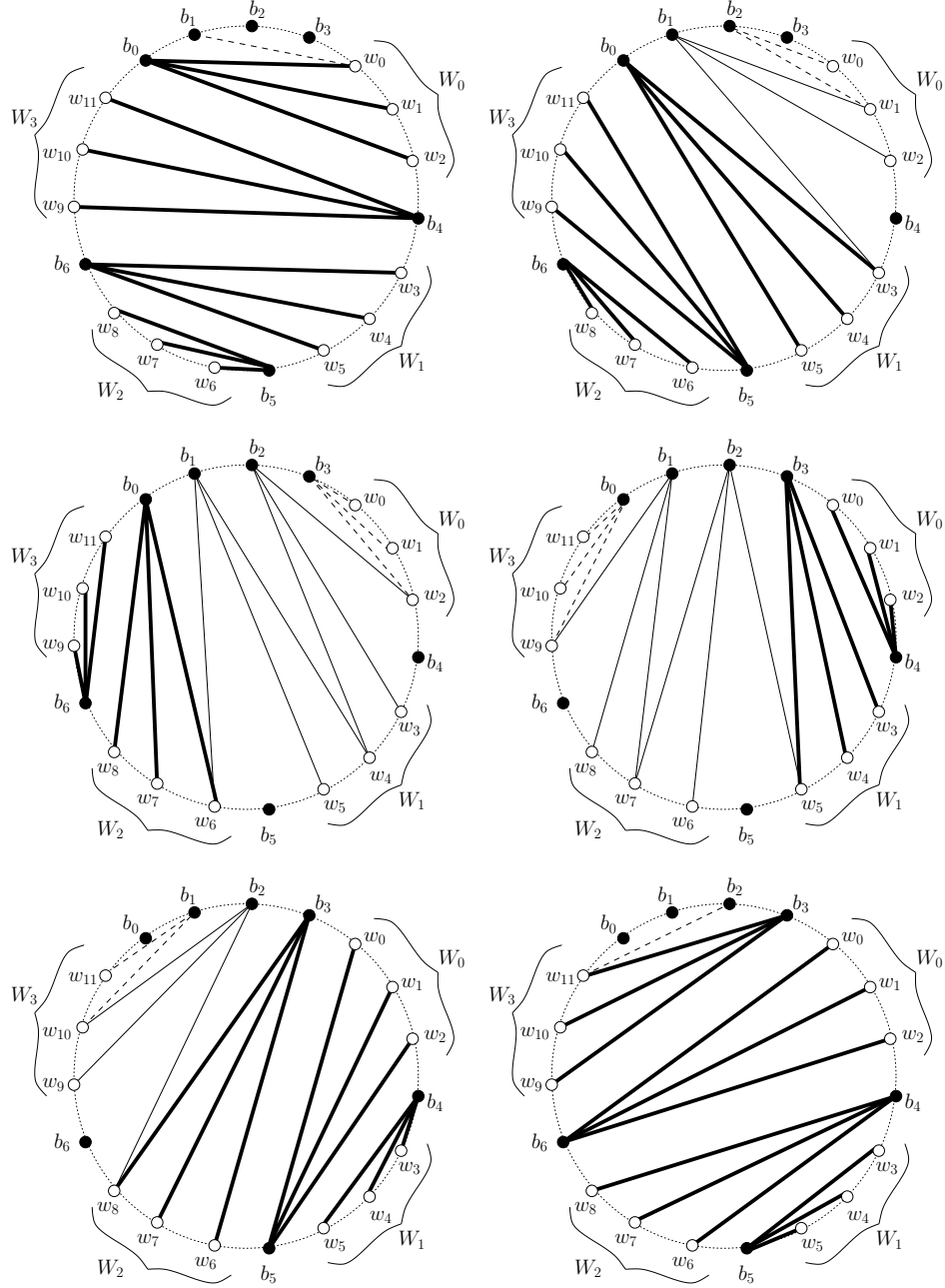


Figure 5: A balanced 6-page embedding of  $K_{7,12}$ . In this case  $k = 6$ , and so  $s = 3$  and  $t = 4$ . Pages 0, 1, 2, 3, 4, and 5 are the upper left, upper right, middle left, middle right, lower left, and lower right circles, respectively. For Pages 0, 1, and 2, we have edges of Types I, II, and III, whereas for Pages 3, 4, and 5, we have edges of Types IV, V, and VI. Edges of Types I and IV are drawn with thick segments; edges of Types II and V are drawn with thinner segments; and edges of Types III and VI are drawn with dashed segments.

400 *Proof of Theorem 6.* For simplicity, we color the  $m$  vertices black, and the  $n$  vertices white.  
 401 Let  $p, q, r, s$  be the nonnegative integers defined by the conditions  $m = kp + r$  and  $0 \leq r \leq$   
 402  $k - 1$ , and  $n = kq + s$  and  $0 \leq s \leq k - 1$  (note that the definitions of  $r$  and  $s$  coincide  
 403 with those in the statement of Theorem 6). Our task is to describe a drawing of  $K_{m,n}$  with  
 404 exactly  $(m - r)(n - s)(m - k + r)(n - k + s)/(4k^2)$  crossings.

405 We start our construction by dividing the set of black vertices into  $k$  groups  $B_0, B_1, \dots, B_{k-1}$ ,  
 406 so that  $k - r$  of them (say the first  $k - r$ ) have size  $p$ , and the remaining  $r$  have size  $p + 1$ .  
 407 Then we divide the set of white vertices into  $k$  groups  $W_0, W_1, \dots, W_{k-1}$ , such that  $k - s$   
 408 of them (say the first  $k - s$ ) have size  $q$ , and the remaining  $s$  have size  $q + 1$ .

409 Then (using the circular drawing model) we place the groups alternately on a circumference,  
 410 as in  $B_0, W_0, B_1, W_1, \dots, B_{k-1}, W_{k-1}$ . Now for  $i = 0, 1, 2, \dots, k - 1$ , we draw in page  $i$  the  
 411 edges joining all black points in  $B_j$  to all white points in  $W_s$  if and only if  $j + s = i$   
 412 (operations are modulo  $k$ ).

413 A straightforward calculation shows that the total number of crossings in this drawing is  
 414  $(k - r)(k - s)\binom{p}{2}\binom{q}{2} + (k - r)s\binom{p}{2}\binom{q+1}{2} + r(k - s)\binom{p+1}{2}\binom{q}{2} + rs\binom{p+1}{2}\binom{q+1}{2}$ , and an elementary  
 415 manipulation shows that this equals  $(m - r)(n - s)(m - k + r)(n - k + s)/(4k^2)$ . Thus  
 416  $\nu_k(K_{m,n}) \leq (m - r)(n - s)(m - k + r)(n - k + s)/(4k^2)$ , as claimed.

417 Finally, note that since obviously  $m - r \leq m$ ,  $n - s \leq n$ ,  $m - k + r \leq m - 1$ , and  
 418  $n - k + s \leq n - 1$ , it follows that  $\nu_k(K_{m,n}) \leq (1/4k^2)m(m - 1)n(n - 1) = (1/k^2)\binom{m}{2}\binom{n}{2}$ .  $\square$

## 419 8 Concluding remarks

420 It seems worth gathering in a single expression the best lower and upper bounds we now  
 421 have for  $\nu_k(K_{m,n})$ . Since  $\nu_k(K_{m,n})$  may exhibit an exceptional behaviour for small values of  
 422  $m$  and  $n$ , it makes sense to express the asymptotic forms of these bounds. The lower bound  
 423 (coming from [22, Theorem 5]) is given in (1), whereas the upper bound is from Theorem 6.

$$\frac{1}{3(3\lceil \frac{k}{2} \rceil - 1)^2} \leq \lim_{m,n \rightarrow \infty} \frac{\nu_k(K_{m,n})}{\binom{m}{2}\binom{n}{2}} \leq \frac{1}{k^2}. \quad (3)$$

424 As we have observed (and used) above,  $\text{cr}_{k/2}(K_{m,n}) \leq \nu_k(K_{m,n})$ , and it is natural to ask  
 425 whether  $\nu_k(K_{m,n})$  is strictly greater than  $\text{cr}_{k/2}(K_{m,n})$  (we assume  $k$  even in this discussion).  
 426 At least in principle, there is much more freedom in  $k/2$ -planar drawings than in  $k$ -page  
 427 drawings. Thus remains the question: can this additional freedom be used to (substantially)  
 428 save crossings?

429 With this last question in mind, we now carry over an exercise which reveals the connections  
 430 between the book and multiplanar crossing numbers of complete and complete bipartite  
 431 graphs.

It is not difficult to prove that the constants

$$\begin{aligned}\text{BOOKBIPARTITE} &:= \lim_{k \rightarrow \infty} k^2 \cdot \left( \lim_{m, n \rightarrow \infty} \frac{\nu_k(K_{m,n})}{\binom{m}{2} \binom{n}{2}} \right), \\ \text{BOOKCOMPLETE} &:= \lim_{k \rightarrow \infty} k^2 \cdot \left( \lim_{n \rightarrow \infty} \frac{\nu_k(K_n)}{\binom{n}{4}} \right), \\ \text{MULTIPLANARBIPARTITE} &:= \lim_{k \rightarrow \infty} k^2 \cdot \left( \lim_{m, n \rightarrow \infty} \frac{\text{cr}_k(K_{m,n})}{\binom{m}{2} \binom{n}{2}} \right), \\ \text{MULTIPLANARCOMPLETE} &:= \lim_{k \rightarrow \infty} k^2 \cdot \left( \lim_{n \rightarrow \infty} \frac{\text{cr}_k(K_n)}{\binom{n}{4}} \right),\end{aligned}$$

are all well-defined.

In view of (3), we have

$$\frac{4}{27} \leq \text{BOOKBIPARTITE} \leq 1. \quad (4)$$

Using the best known upper bound for  $\text{cr}_k(K_{m,n})$  (from [22, Theorem 8]), we obtain

$$\text{MULTIPLANARBIPARTITE} \leq \frac{1}{4}. \quad (5)$$

We also invoke the upper bound  $\text{cr}_k(K_n) \leq (1/64)k(n+k^2)^4/(k-1)^3$ , which holds whenever  $k$  is a power of a prime and  $n \geq (k-1)^2$  (see [22, Theorem 7]). This immediately yields

$$\text{MULTIPLANARCOMPLETE} \leq \frac{3}{8}. \quad (6)$$

We also note that the observation  $\text{cr}_{k/2}(K_{m,n}) \leq \nu_k(K_{m,n})$  immediately implies that

$$\text{MULTIPLANARBIPARTITE} \leq \frac{1}{4} \cdot \text{BOOKBIPARTITE}. \quad (7)$$

Finally, applying the Richter-Thomassen counting argument [19] for bounding the crossing number of  $K_{2n}$  in terms of the crossing number of  $K_{n,n}$  (their argument applies unmodified to  $k$ -planar crossing numbers), we obtain

$$\text{MULTIPLANARCOMPLETE} \geq \frac{3}{2} \cdot \text{MULTIPLANARBIPARTITE}. \quad (8)$$

Suppose that the multiplanar drawings of Shahrokhi et al. [22] are asymptotically optimal. In other words, suppose that equality holds in (5). Using (7), we obtain  $\text{BOOKBIPARTITE} \geq 1$ , and by (4) then we get  $\text{BOOKBIPARTITE} = 1$ . Moreover (again, assuming equality holds in (5)), using (8), we obtain  $\text{MULTIPLANARCOMPLETE} \geq 3/8$ , and so in view of (6) we get  $\text{MULTIPLANARCOMPLETE} = 3/8$ . Summarizing:

**Observation 21.** Suppose that the multiplanar drawings of  $K_{m,n}$  of Shahrokhi et al. [22] are asymptotically optimal, so that  $\text{MULTIPLANARBIPARTITE} = \frac{1}{4}$ . Then

$$\begin{aligned} \text{BOOKBIPARTITE} &= 1, \quad \text{and} \\ \text{MULTIPLANARCOMPLETE} &= \frac{3}{8}. \end{aligned} \quad \square$$

In other words, under this scenario (the multiplanar drawings of  $K_{m,n}$  in [22] being asymptotically optimal), the additional freedom of  $(k/2)$ -planar over  $k$ -page drawings of  $K_{m,n}$  becomes less and less important as the number of planes and pages grows. In addition, under this scenario the  $k$ -planar crossing number of  $K_n$  also gets (asymptotically) determined.

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